Embedded Systems
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Basilica di Santa Maria di Collemaggio, 1287, L'Aquila

Lecture 5
Review of Lyapunov Stability Theory

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Outline

- Stability of equilibrium points
- Lyapunov Stability
- Examples

Acknowledgement: H. K. Khalil, ***Nonlinear Systems*** (3rd Edition), Prentice Hall, 2002.

Nonlinear state model

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
&\vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{aligned}
$$

$\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t

$u_1, u_2, \ldots, u_p$ are input variables

$x_1, x_2, \ldots, x_n$ the state variables

Nonlinear state model

$$
\begin{aligned}
\dot{x} &= f(t, x, u) \\
y &= h(t, x, u)
\end{aligned}
$$

x is the state, u is the input
y is the output (q-dimensional vector)

Special cases

Linear systems $$\dot{x} = A(t)x + B(t)u$$

Autonomous systems $$\dot{x} = f(t, x)$$

Time-invariant systems $$\dot{x} = f(x, u)$$

Linear Time-invariant (LTI) systems $$\dot{x} = Ax + Bu$$

Existence and uniqueness of solutions

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x over the domain of interest

$f(t, x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities

$f(t, x)$ is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0, r) = \{x \in R^n \mid \|x - x_0\| < r\}$ where $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad L > 0$$

Local existence and uniqueness of solutions

$$\dot{x} = f(t, x)$$

A function $f(t, x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$

Example: $f(x) = -x^2$ is locally Lipschitz for all x

Global existence and uniqueness of solutions

$$\dot{x} = f(t, x)$$

A function $f(t, x)$ is globally Lipschitz in x if

$$\|f(t, x) - f(t, y)\| \; \leq \; L\|x - y\|$$

for all $x, y \in R^n$ with the same Lipschitz constant L

Lemma: Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$

Example: $f(x) = -x^2$ is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded

Nonlinear phenomena

- ✓ Finite escape time
- ✓ Multiple isolated equilibrium points
- ✓ Limit cycles
- ✓ Subharmonic, harmonic, or almost-periodic oscillations
- ✓ Chaos
- ✓ Multiple modes of behavior

Linearization

A common engineering practice in analyzing a nonlinear system is to linearize it about some nominal operating point and analyze the resulting linear model

What are the limitations of linearization?

- Since linearization is an approximation in the neighborhood of an operating point, it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the "nonlocal" or "global" behavior
- There are "essentially nonlinear phenomena" that can take place only in the presence of nonlinearity

Outline

- Introduction: nonlinear models
- Stability of equilibrium points
- Lyapunov Stability
- Examples

Equilibrium points

A point $x = x^*$ in the state space is said to be an equilibrium point of $\dot{x} = f(t, x)$ if

$$x(t_0) = x^* \;\Rightarrow\; x(t) \equiv x^*, \;\; \forall\, t \geq t_0$$

For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real solutions of the equation

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points

Equilibrium points

A linear system $\dot{x} = Ax$ can have an isolated equilibrium point at $x = 0$ (if A is nonsingular) or a continuum of equilibrium points in the null space of A (if A is singular)

It cannot have multiple isolated equilibrium points , for if x_a and x_b are two equilibrium points, then by linearity any point on the line $\alpha x_a + (1-\alpha)x_b$ connecting x_a and x_b will be an equilibrium point

A nonlinear state equation can have multiple isolated equilibrium points .For example, the state equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 - b x_2$$

has equilibrium points at $(x_1 = n\pi, x_2 = 0)$ for $n = 0, \pm 1, \pm 2, \cdots$

Equilibrium points

$$\dot{x} = f(x)$$

f is locally Lipschitz over a domain $D \subset R^n$

Suppose $\bar{x} \in D$ is an equilibrium point; that is, $f(\bar{x}) = 0$

Characterize and study the stability of $\bar{x}$

For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. No loss of generality

$$y = x - \bar{x}$$

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

Stability of equilibrium points

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- stable if for each $\varepsilon > 0$ there is $\delta > 0$ (dependent on ε) such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq 0$$

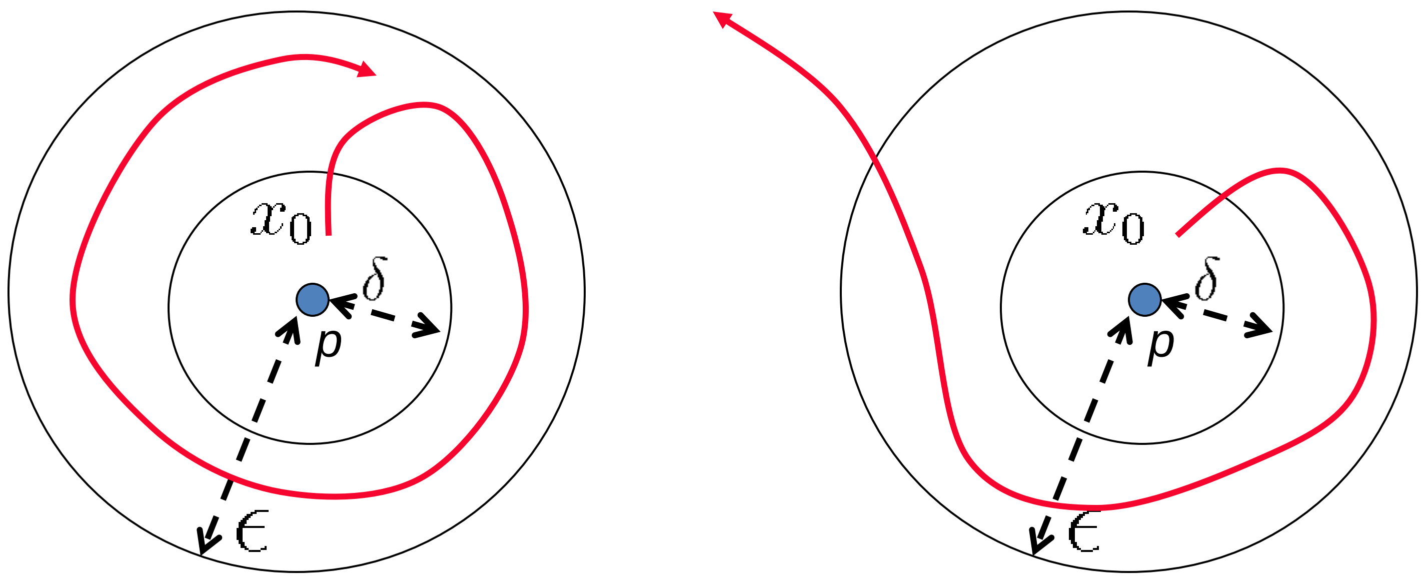
- unstable if it is not stable

- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

Asymptotic Stability

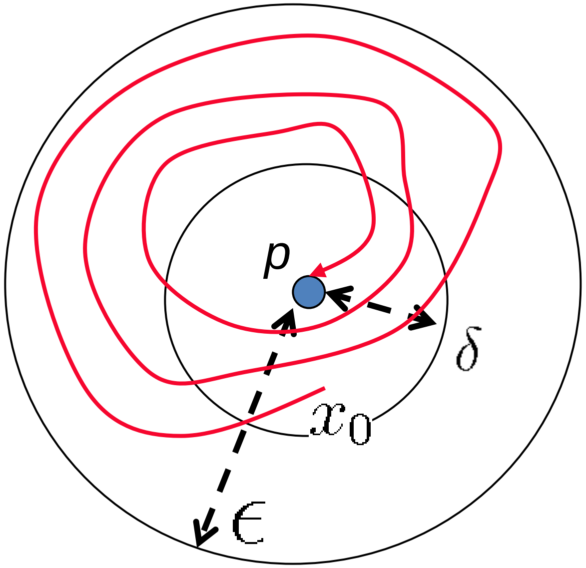
- Given $\dot{x} = f(x)$, let p be an equilibrium, i.e. $f(p) = 0$.
- The equilibrium p is stable if for any $\epsilon > 0$, there is a $\delta(\epsilon)$, such that the trajectory with initial condition x_0, with $\|x_0 - p\| < \delta(\epsilon)$ remains within ϵ distance from p.



stable

Asymptotic Stability

- The equilibrium p is asymptotically stable if for any $\epsilon > 0$, there is a $\delta(\epsilon)$, such that the trajectory with initial condition x_0, with $\|x_0 - p\| < \delta(\epsilon)$ remains within ϵ distance from p and converge to p.



Asymptotically stable

Region of attraction and global stability

Definition: Let the origin be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where f is a locally Lipschitz function defined over a domain $D \subset R^n$ ($0 \in D$)

- The region of attraction (also called region of asymptotic stability, domain of attraction, or basin) is the set of all points x_0 in D such that the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

  is defined for all $t \geq 0$ and converges to the origin as t tends to infinity

- The origin is said to be globally asymptotically stable if the region of attraction is the whole space R^n

Stability of linear time invariant systems

- A system is stable if with zero input, starting from any initial condition, the state trajectory converges to zero.
$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{At} x(0) = 0.$$
- $\mathcal{L}(e^{At}) = (sI - A)^{-1}$. The polynomial $\det(sI - A)$ is called the characteristic polynomial.
- The system is stable if and only if all the roots of the characteristic polynomial have negative real part.
- Stability also implies that bounded input will produce bounded output.

Stability of LTI systems

Theorem: The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \leq 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$

When all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$, A is called a *Hurwitz matrix*

When the origin of a linear system is asymptotically stable, its solution satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0$$

$k \geq 1, \;\lambda > 0$

Exponential Stability

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is said to be exponentially stable if

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0$$

$k \geq 1,\ \lambda > 0$, for all $\|x(0)\| < c$

It is said to be globally exponentially stable if the inequality is satisfied for any initial state $x(0)$

Exponential Stability $\Rightarrow$ Asymptotic Stability

Linearization

$$\dot{x} = f(x), \quad f(0) = 0$$

f is continuously differentiable over $D = \{\|x\| < r\}$

$$J(x) = \frac{\partial f}{\partial x}(x)$$

Set $A = J(0)$. In a small neighborhood of the origin we can approximate the nonlinear system $\dot{x} = f(x)$ by its linearization about the origin $\dot{x} = Ax$

Theorem:
- The origin is exponentially stable if and only if $\mathrm{Re}[\lambda_i] < 0$ for all eigenvalues of A
- The origin is unstable if $\mathrm{Re}[\lambda_i] > 0$ for some i

Linearization fails when $\mathrm{Re}[\lambda_i] \leq 0$ for all i, with $\mathrm{Re}[\lambda_i] = 0$ for some i

Linearization: example

Theorem:

- The origin is exponentially stable if and only if $\mathrm{Re}[\lambda_i] < 0$ for all eigenvalues of A
- The origin is unstable if $\mathrm{Re}[\lambda_i] > 0$ for some i

Example

$$\dot{x} = ax^3$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\big|_{x=0} = 0$$

Stable if $a = 0$; Asymp stable if $a < 0$; Unstable if $a > 0$

When $a < 0$, the origin is not exponentially stable

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Lyapunov Stability

Let $V(x)$ be a continuously differentiable function defined in a domain $D \subset R^n$; $0 \in D$. The derivative of V along the trajectories of $\dot{x} = f(x)$ is

$$\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\
&= \left[\frac{\partial V}{\partial x_1}, \ \frac{\partial V}{\partial x_2}, \ \ldots, \ \frac{\partial V}{\partial x_n} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \\
&= \frac{\partial V}{\partial x} f(x)
\end{aligned}$$

Lyapunov Stability Theorem

- If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \in D/\{0\}$$

$$\dot{V}(x) \leq 0, \quad \forall\, x \in D$$

then the origin is stable

- Moreover, if

$$\dot{V}(x) < 0, \quad \forall\, x \in D/\{0\}$$

then the origin is asymptotically stable

- Furthermore, if $V(x) > 0, \forall\, x \neq 0,$

$$\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty$$

and $\dot{V}(x) < 0, \forall\, x \neq 0,$ then the origin is globally asymptotically stable

Lyapunov Stability Theorem

Terminology

$V(0) = 0,\ V(x) \geq 0$ for $x \neq 0$	Positive semidefinite
$V(0) = 0,\ V(x) > 0$ for $x \neq 0$	Positive definite
$V(0) = 0,\ V(x) \leq 0$ for $x \neq 0$	Negative semidefinite
$V(0) = 0,\ V(x) < 0$ for $x \neq 0$	Negative definite
$\lVert x \rVert \to \infty \;\Rightarrow\; V(x) \to \infty$	Radially unbounded

Lyapunov Theorem: The origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite. It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded

Lyapunov Functions

A continuously differentiable function $V(x)$ satisfying the conditions for stability is called a *Lyapunov function*.

Quadratic Lyapunov Functions

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \quad P = P^T$$

$$\lambda_{\min}(P) \|x\|^2 \leq x^T P x \leq \lambda_{\max}(P) \|x\|^2$$

$P \geq 0$ (Positive semidefinite) if and only if $\lambda_i(P) \geq 0 \; \forall i$

$P > 0$ (Positive definite) if and only if $\lambda_i(P) > 0 \; \forall i$

$V(x)$ is positive definite if and only if P is positive definite
$V(x)$ is positive semidefinite if and only if P is positive semidefinite
$P > 0$ if and only if all the leading principal minors of P are positive

Lyapunov Functions for LTI systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x, \qquad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x \stackrel{\text{def}}{=} -x^T Q x$$

If $Q > 0$, then A is Hurwitz

Or choose $Q > 0$ and solve the Lyapunov equation

$$PA + A^T P = -Q$$

If $P > 0$, then A is Hurwitz

Lyapunov Theorem for LTI systems

Theorem A matrix A is Hurwitz if and only if for any $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

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Example 1

$$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= \quad x_1 + (x_1^2 - 1)x_2\end{aligned}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

has eigenvalues $(-1 \pm j\sqrt{3})/2$. Hence the origin is asymptotically stable

Example 2: pendulum equations without friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \tfrac{1}{2}x_2^2$$

$V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0$$

The origin is stable

Since $\dot{V}(x) \equiv 0$, the origin is not asymptotically stable

Example 3: pendulum equations with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2\end{aligned}$$

$$V(x) = a(1 - \cos x_1) + \frac{1}{2} x_2^2$$

$$\dot{V}(x) = a \dot{x}_1 \sin x_1 + x_2 \dot{x}_2 = -b x_2^2$$

The origin is stable

$\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

Example 3: pendulum equations with friction

The conditions of Lyapunov's theorem are only sufficient. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate

Try

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T P x + a(1-\cos x_1) \\ &= \tfrac{1}{2}[x_1\ x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1-\cos x_1) \end{aligned}$$

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

Stability of nonlinear systems

In the following we focus on the following stability notions:

- Global Asymptotic Stability (GAS)
- Input-to-State Stability (ISS)
- Incremental Global Asymptotic Stability (δ-GAS)
- Incremental Input-to-State Stability (δ-ISS)

Stability of nonlinear systems

1- A continuous function α: $[0,a) \rightarrow [0,\infty)$ is said to be a class K function if it is strictly increasing and $\alpha(0) = 0$. Function α: $[0,\infty) \rightarrow [0,\infty)$ is said to be a class K_∞ if it is a class K function and $\lim_{r\rightarrow\infty} \alpha(r) = \infty$.

2- A continuous function β: $[0,a) \times [0,\infty) \rightarrow [0,\infty)$ is said to be a class KL function if for each fixed s, function $\beta(r,s)$ is a class K function and for each fixed r, function $\beta(r,s)$ is decreasing and tends to zero as s goes to ∞.

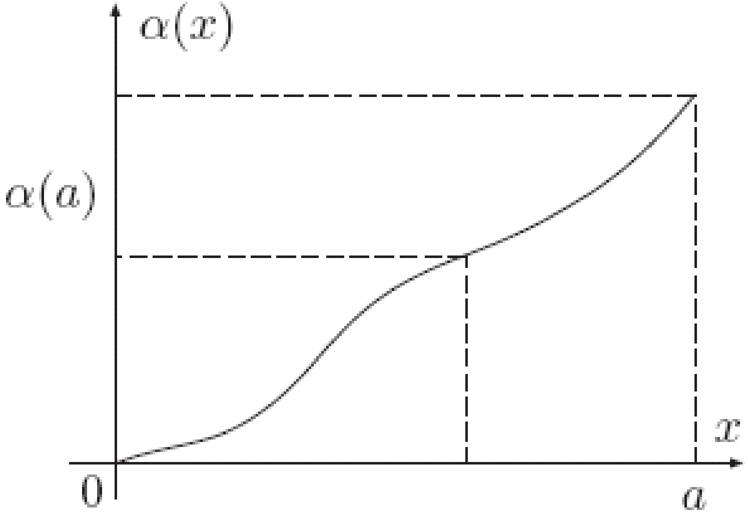


Figura 8 – Funzione di classe K

Global Asymptotic Stability

The equilibrium x=0 of $\dot{x} = f(x,u)$ is Globally Asymptotically Stable (GAS) if there exists a KL function β so that for any $t \geq 0$, $y \in R^n$ and **u = 0**

$\| \mathbf{x}(t,y,0) \| \leq \beta(\|y\|, t)$

Theorem:

The equilibrium x=0 of $\dot{x} = f(x,u)$ is GAS if there exists a smooth function $V : R^n \to R^+$ such that:

i) $V(x) \geq 0$ for all $x \in R^n$ and $V(x) = 0$ if and only if $x = 0$

ii) $dV/dx\ f(x,0) < 0$ for all $x \in R^n$

Further details from H. K. Khalil, Nonlinear Systems (3rd Edition), Prentice Hall, 2002

Input-to-State Stability

A control system $\dot{x} = f(x,u)$ is Input-to-State Stable (ISS) if there exist a KL function $\boldsymbol{\beta}$ and a K_∞ function γ so that for any $t \geq 0$, $y \in R^n$ and $\mathbf{u}$

$$\| \mathbf{x}(t,y,\mathbf{u}) \| \leq \beta(\|y\| , t) + \gamma(\|\mathbf{u}\|_\infty)$$

Theorem:
A control system $\dot{x} = f(x,u)$ is ISS if there exists a smooth function $V : R^n \rightarrow R^+$ and K_∞ functions α_1, α_2, ρ, σ such that:

i) $\alpha_1 (\|x\|) \leq V(x) \leq \alpha_2 (\|x\|)$ for all $x \in R^n$

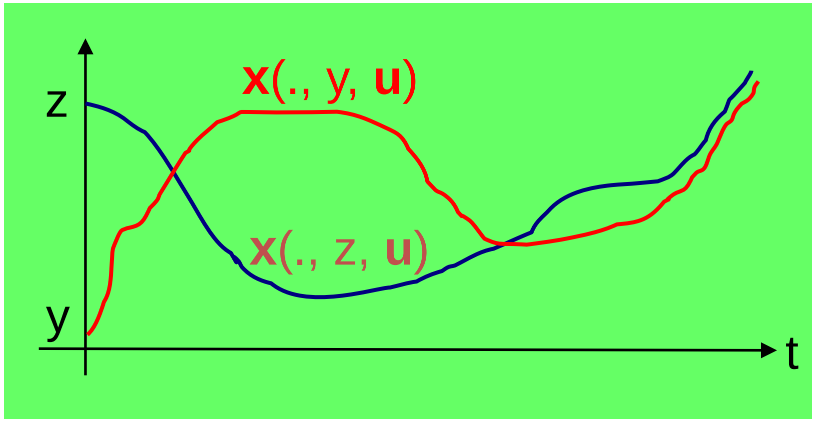
ii) $dV/dx\, f(x,\mathbf{u}) < - \rho (\|x\|) + \sigma (\|u\|)$ for all $x \in R^n$ and $u \in R^m$

Further details from H. K. Khalil, Nonlinear Systems (3rd Edition), Prentice Hall, 2002

Incremental Global Asymptotic Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Globally Asymptotically Stable (δ-GAS) if there exists a KL function β so that for any $t \geq 0$, $y,z \in R^n$ and $\mathbf{u}$

$$\| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{u}) \| \leq \beta(\|y-z\|, t)$$



Additional details from D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Incremental Global Asymptotic Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Globally Asymptotically Stable (δ-GAS) if there exists a KL function **β** so that for any $t \geq 0$, $y,z \in R^n$ and **u**

$\| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{u}) \| \leq \boldsymbol{\beta}(\|y-z\|, t)$

Theorem:
A control system $\dot{x} = f(x,u)$ is δ-GAS if there exists a smooth function $V : R^n \times R^n \rightarrow R^+$ and K_∞ functions α_1, α_2, ρ such that:

i) $\alpha_1 (\| x - y \|) \leq V(x, y) \leq \alpha_2 (\| x - y \|)$ for all $x,y \in R^n$

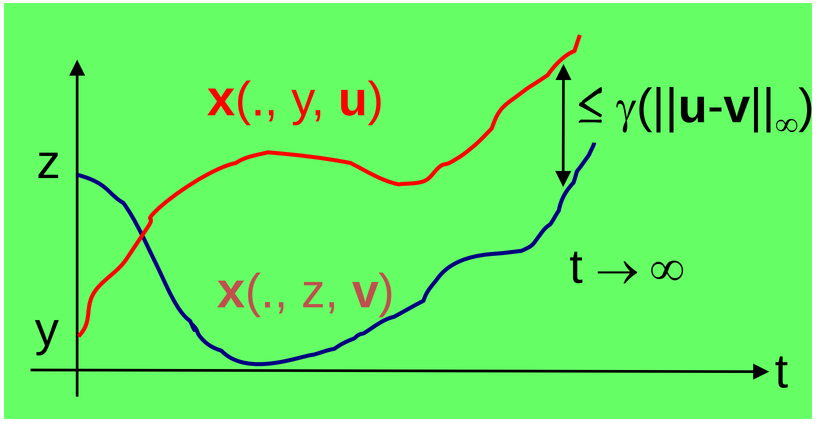
ii) $dV/dx\, f(x,u) + dV/dy\, f(y,u) < - \rho (\| x - y \|)$ for all $x,y \in R^n$ and $u \in R^m$

Additional details in D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Incremental Input-State Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Input-to-State Stable (δ-ISS) if there exist a KL function $\boldsymbol{\beta}$ and a K_∞ function γ so that for any $t \geq 0$, $y, z \in R^n$ and **u**, **v**

$$\| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{v}) \| \leq \boldsymbol{\beta}(\|y - z\|, t) + \gamma(\|\mathbf{u} - \mathbf{v}\|_\infty)$$



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Incremental Input-State Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Input-to-State Stable (δ-ISS) if there exist a KL function $\boldsymbol{\beta}$ and a K_∞ function γ so that for any $t \geq 0$, $y, z \in R^n$ and $\mathbf{u}$, $\mathbf{v}$

$$|| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{v}) || \leq \boldsymbol{\beta}(||y - z||, t) + \gamma(||\mathbf{u} - \mathbf{v}||_\infty)$$

Theorem:

A control system $\dot{x} = f(x,u)$ is δ-ISS if there exists a smooth function $V : R^n \times R^n \to R^+$ and K_∞ functions α_1, α_2, ρ, σ such that:

i) $\alpha_1 (||x - y||) \leq V(x, y) \leq \alpha_2 (||x - y||)$ for all $x,y \in R^n$

ii) $dV/dx\, f(x,u) + dV/dy\, f(y,v) < - \rho (||x - y||) + \sigma (||u - v||)$ for all $x,y \in R^n$ and $u,v \in R^m$

Further details from D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Connections among Stability Notions

$$
\begin{tikzcd}
\delta\text{-ISS} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "f(0,0)=0"'] & \delta\text{-GAS} \arrow[d, Rightarrow, "f(0,0)=0"] \\
\text{ISS} \arrow[r, Rightarrow] & \text{GAS}
\end{tikzcd}
$$

Homework:

1) Prove such connections!
2) How do these notions specialize to the case of linear control systems?